Name Joud Hamdan , ID 95608

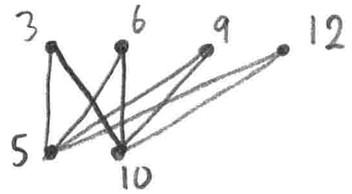
MTH 213, Final

Ayman Badawi

$$Score = \frac{62}{62}$$ excellent

QUESTION 1. (6 points) Let $G(V, E)$ be a graph of order 6, where $V = \{3, 5, 6, 9, 10, 12\}$. For every $a, b \in V$, $a - b$ is an edge of G iff $(ab)\ (mod\ 15) = 0$.

(i) By drawing the graph, convince me that G is a complete bipartite graph.



(ii) Is G Hamiltonian? if no, explain. If yes, construct such Hamiltonian cycle.

No, C_6 is not a subgragh

(iii) Is G an Eulerian? if no, explain. If yes, construct an Eulerian circuit.

Yes since deg(every vertex) is even

3 - 10 - 12 - 5 - 6 - 10 - 9 - 5 - 3

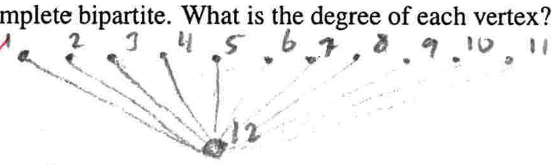
QUESTION 2. (6 points) Let G be a connected graph.

(i) Assume G is of order 2023 and size 2022. Let a, b be two vertices of G. How many paths are there between a, b? Explain. 1 path.

Since $|E| = |V| - 1$, G is a tree & G is a tree iff $\exists!$ path between every 2 vertices.

(ii) Assume G is order 12 and size 11. Given that G is a complete bipartite. What is the degree of each vertex?

$|V| = 12$ $|E| = 11$



deg (vertex 1 to 11) = 1 and deg(vertex 12) = 11

(iii) Assume G is complete of order 104. How many edges does G have? i.e., what is the size of G?

Between every 2 vertices, there is an edge ∴

$_{104}C_2$

QUESTION 3. (6 points) Let $G(V,E)$ be a graph of order 5, where $V = \{2,3,4,9,15\}$. For every two vertices $a, b \in V$, $a - b$ is an edge iff $(ab)\ (mod\ 6) = 0$.

(i) Is G bipartite? If yes, draw it.

Yes

(ii) Convince me that G is neither Hamiltonian nor Eulerian?

(G is bipartite iff it has no odd cycles)

- It can't be Hamiltonian since it's bipartite and C_5 cannot be a subgraph
- Not Eulerian as deg(every vertex) is not even

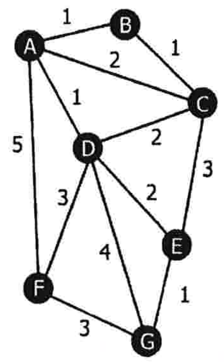
(iii) By contruction of trail(path), convince me that G is an Euler trail and a Hamilton path.

Euler trail: 2 - 9 - 4 - 3 - 2 - 15 - 4

Hamiltonian path: 15 - 4 - 3 - 2 - 9

QUESTION 4. (6 points) Stare at the below graph. Use Dijkstra's Algorithm (as explained in class) and construct the minimum spanning tree, i.e., finding the minimum weighted path between every two vertices.

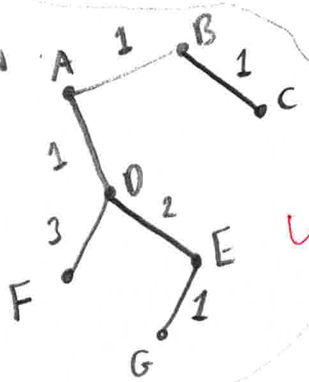
[redacted]



[redacted]

	A	B	C	D	E	F	G
A	$\boxed{0}$	1^A	2^A	1^A	∞	5^A	∞
B		$\boxed{1}^A$	2^B	1^A	∞	5^A	∞
D			2^B	$\boxed{1}^A$	3^D	4^D	5^D
C			$\boxed{2}^B$		3^D	4^D	5^D
E					$\boxed{3}^D$	4^D	4^E
F						$\boxed{4}^D$	4^E
G							$\boxed{4}^E$

QUESTION 5. (6 points) Let x be your score on an exam out of 77, i.e., $0 \le x < 77$. Given $x\ (mod\ 7) = 3$ and $x\ (mod\ 11) = 7$. Use the CRT and find the value of x.



$m_1 = 7$, $m_2 = 11$, $m = 7(11) = 77$

$n_1 = \frac{m}{m_1} = 11$, $n_2 = \frac{m}{m_2} = 7$

$11 y_1 = 1$ in Z_7 $\qquad$ $7 y_2 = 1$ in Z_{11}

$4 y_1 = 1$ in Z_7 $\qquad$ $y_2 = 8$

$y_1 = 2$

$x = (3(11)(2) + 7(7)(8))\ (mod\ 77)$

$= 458\ (mod\ 77) = 73$

QUESTION 6. (8 points)

(i) The digits 0, 1, 2, 3, ..., 9 are used to construct 8-digits ID-cards (note that 10 digits are available, i.e., from 0 to 9). If repetition is allowed, exactly 3 digits are 2, and exactly two digits are 5; how many ID-cards can be constructed?

$_8C_3 \times {_5C_2} \times 10^3$

(ii) Out of 21 available persons (11 males and 10 females), a committee with 9 persons is formed. Assume that $f_1, f_2, ..., f_{10}$ are the names of the females and $m_1, m_2, ..., m_{11}$ are the names of the males. If f_8, f_9, f_{10}, and exactly 4 males must be in the committee, in how many different ways can we form such a committee?

$_{11}C_4 \times {_7C_2}$

(iii) What is the minimum number of chocolate bags that can be distributed over 32 schools so that a school will have at least 19 bags of chocolate?

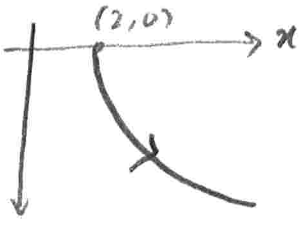
$\left\lceil \frac{n}{32} \right\rceil = 19 \Rightarrow n = 32 \times 18 + 1 = 577$

(iv) There are 920 positive integers such that each integer is of form 6k for some integer $k \in Z$. Then there are at least m integers out of the 920 numbers, say, $n_1, \ldots, n_m$ such that $n_1 \ (mod\ 9) = n_2 \ (mod\ 9) = \cdots = n_m \ (mod\ 9)$. What is the best value of m?

n
920 integers
C
0 3 6

$m = \left\lceil \frac{920}{3} \right\rceil = 307$

QUESTION 7. (6 points) Let $f : [2, \infty[\ \rightarrow \ ]-\infty, 0]$ be a function such that $f(x) = -\sqrt{x-2}$.



(i) By drawing, is f one-to-one and onto?

Yes

(ii) If the answer to (i) is yes, find the domain and the co-domain of $f^{-1}(x)$, then find $f^{-1}(x)$.

$f^{-1} :]-\infty, 0] \rightarrow [2, \infty[$

$x = -\sqrt{y-2}$

$x^2 = y - 2 \Leftrightarrow y = x^2 + 2$

$f^{-1}(x) = x^2 + 2$

QUESTION 8. (6 points) Use math induction and prove that $14 \mid (13^{(2n)} - 1)$ for every positive integer $n \geq 1$.

① Show it's true for $n = 1$: $13^2 - 1 = 168 = 14(12) \Rightarrow 14$ is a factor of $13^2 - 1$

② Assume $14 \mid (13^{2n} - 1)$ for some $n \geq 1$

③ Show $14 \mid (13^{2n+2} - 1)$

$$\begin{aligned} 13^{2n+2} - 1 &= 13^{2n} \cdot 13^2 - 1 \\ &= 13^{2n} \cdot 13^2 - 1 + 13^2 - 13^2 \\ &= 13^2(13^{2n} - 1) + 13^2 - 1 \\ &= 13^2(14k) + 12(14) \text{ for some } k \geq 1 \\ &= 14(13^2 k + 12). \text{ Done.} \end{aligned}$$

QUESTION 9. (6 points)

(i) Let $A = \{2,4,5,7\}$. Define " = " on A such that for every $a, b \in A$, $a" = "b$ iff $b = ak$ for some integer $k \neq 0$. Convince me that " = " is not an equivalence relation.

$a = 2$, $b = 4$ and $2 \in Z^*$

$a" = "b$ since $b = 2a$ but $b" \neq "a$ since $a = \frac{1}{2}b$ and $\frac{1}{2} \notin Z^*$

$\therefore$ Symmetric axiom fails and "=" is not an equivalence relation on A

(ii) Let $A = \{2, -2, 7, -7, 9, -9, 11\}$. Define " = " on A such that for every $a, b \in A$, $a" = "b$ iff $b = ak$ for some integer $k \neq 0$. Then " = " is an equivalence relation. Find all distinct equivalence classes.

$[2] = \{2, -2\}$ $[9] = \{9, -9\}$

$[7] = \{7, -7\}$ $[11] = \{11\}$

QUESTION 10. (6 points) Let $a_n = 2a_{n-1} + 35a_{n-2} + 24(3^n)$. Find a general formula for a_n, no need to find c_1, c_2.

$$a_n - 2a_{n-1} - 35a_{n-2} = 24(3^n)$$

$a_n - 2a_{n-1} - 35a_{n-2} = 0$

$\alpha^n - 2\alpha^{n-1} - 35\alpha^{n-2} = 0$

$\alpha^2 - 2\alpha - 35 = 0$

$(\alpha - 7)(\alpha + 5) = 0$

$\alpha = 7$ or $\alpha = -5$

$a_h(n) = c_1(7^n) + c_2(-5)^n$

$a_p(n) = A(3^n)$

$a_p(n) - 2a_p(n-1) - 35a_p(n-2) = 24(3^n)$

$A(3^n) - 2A(3^{n-1}) - 35A(3^{n-2}) = 24(3^n)$

$A(3^n) - \frac{2}{3}A(3^n) - \frac{35}{9}A(3^n) = 24(3^n)$

$A - \frac{2}{3}A - \frac{35}{9}A = 24 \Rightarrow A = -\frac{27}{4}$

$a_n = a_h(n) + a_p(n) = c_1(7^n) + c_2(-5)^n - \frac{27}{4}(3^n)$

Name ELISHA MARY THOMAS, ID g00092432

MTH 212, Final

Ayman Badawi

$$Score = \frac{62}{62}$$ Excellent

QUESTION 1. (6 points) Let $G(V, E)$ be a graph of order 6, where $V = \{3, 5, 6, 9, 10, 12\}$. For every $a, b \in V$, $a - b$ is an edge of G iff $(ab)\ (mod\ 15) = 0$.

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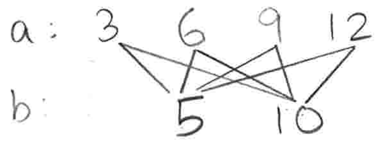
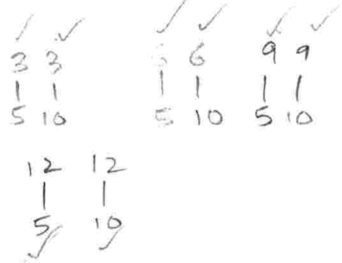


→ Complete bipartite graph



(ii) Is G Hamiltonian? if no, explain. If yes, construct such Hamiltonian cycle.

No, as C_6 is not possible.
For Hamiltonian cycle, we should have a subgraph C_n, where n = order.

Rough

(iii) Is G an Eulerian? if no, explain. If yes, construct an Eulerian circuit.

Yes, G is an Eulerian as degree of each vertex (V) = even number

6 − 10 − 12 − 5 − 9 − 10 − 3 − 5 − 6

QUESTION 2. (6 points) Let G be a connected graph.

(i) Assume G is of order 2023 and size 2022. Let a, b be two vertices of G. How many paths are there between a, b? Explain.

Only one path as this is a tree graph.
Since order is n & size is $(n-1)$.

(ii) Assume G is order 12 and size 11. Given that G is a complete bipartite. What is the degree of each vertex?

verify: degree $= 11 + 11 = 22 = 2|E|$

Let the bipartite be $K_{n,m}$ · $n = |A|$, $m = |B|$

$n + m = 12$
$nm = 11$
$m = \frac{11}{n}$

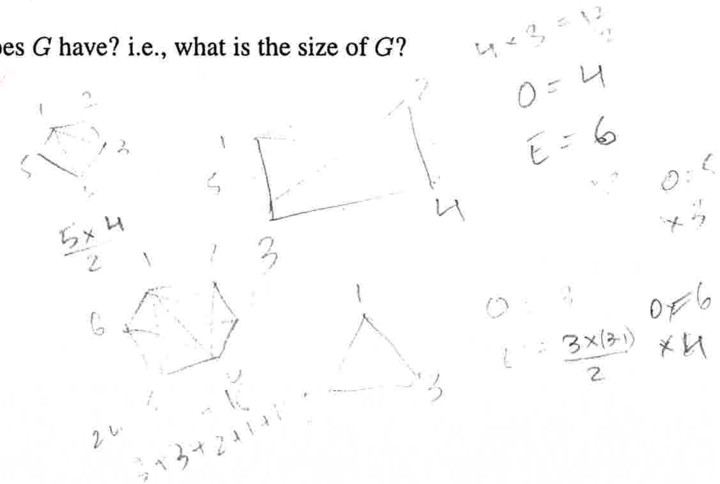
$n + \frac{11}{n} = 12$
$n^2 - 12n + 11 = 0$
$n = 11, 1$

(A) Degree of one vertex is $\boxed{11}$
(B) Degree of the other 11 vertices are $\boxed{1}$ each.

(iii) Assume G is complete of order 104. How many edges does G have? i.e., what is the size of G?

$|E| = \frac{\sum degree}{2}$

Size $= (104 \times 103)/2$
$= 5356$



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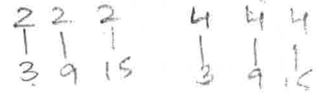
(i) Is G bipartite? If yes, draw it.

Yes, G is bipartite



(ii) Convince me that G is neither Hamiltonian nor Eulerian?

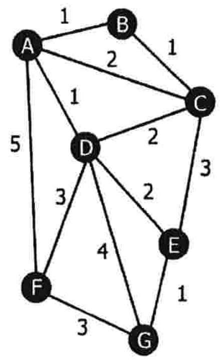
G is not Eulerian → as deg(2) = 3 = deg(4) → two vertices have odd degrees

It is not Hamiltonian → as C_5 is not possible in G since it is bipartite

(iii) By contruction of trail(path), convince me that G is an Euler trail and a Hamilton path.

Euler trail: 2 — 3 — 4 — 9 — 2 — 15 — 4 | Hamilton path: 3 — 2 — 9 — 4 — 15

QUESTION 4. (6 points) Stare at the below graph. Use Dijkstra's Algorithm (as explained in class) and construct the minimum spanning tree, i.e., finding the minimum weighted path between every two vertices.

[redacted]  [redacted]

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D			2^B	$\boxed{1^A}$	3^D	4^D	5^D
C			$\boxed{2^B}$		3^D	4^D	5^D
E					$\boxed{3^D}$	4^D	4^E
F						$\boxed{4^D}$	4^E
G							$\boxed{4^E}$

(Drawn on next page)

QUESTION 5. (6 points) Let x be your score on an exam out of 77, i.e., $0 \le x < 77$. Given $x\ (mod\ 7) = 3$ and $x\ (mod\ 11) = 7$. Use the CRT and find the value of x.

$r_1 = 3$, $r_2 = 7$

$m_1 = 7$, $m_2 = 11$

① $n_1 = \frac{m}{m_1} = 11$, $n_2 = \frac{m}{m_2} = 7$

② $11 y_1 = 1$ in Z_7

$4 y_1 = 1$ in Z_7

$y_1 = 2$

$7 y_1 = 1$ in Z_{11}

$y_1 = 8$

③ $x = [r_1 n_1 y_1 + r_2 n_2 y_2] \bmod m = [3 \times 11 \times 2 + 7 \times 7 \times 8] \bmod 77$

$= 458 \bmod 77$

$\boxed{x = 73}$

QUESTION 6. (8 points)

(i) The digits 0, 1, 2, 3, …, 9 are used to construct 8-digits ID-cards (note that 10 digits are available, i.e., from 0 to 9). If repetition is allowed, exactly 3 digits are 2, and exactly two digits are 5; how many ID-cards can be constructed?

3 two + 2 five

$= {}^{8}C_{3} \cdot {}^{5}C_{2} \cdot 8^{3}$

(ii) Out of 21 available persons (11 males and 10 females), a committee with 9 persons is formed. Assume that $f_1, f_2, ..., f_{10}$ are the names of the females and $m_1, m_2, ..., m_{11}$ are the names of the males. If f_8, f_9, f_{10}, and exactly 4 males must be in the committee, in how many different ways can we form such a committee?

$f_8\ f_9\ f_{10}$ 4M 2F

$= (11C4) \cdot (7C2) = \binom{11}{4}\binom{7}{2}$

(iii) What is the minimum number of chocolate bags that can be distributed over 32 schools so that a school will have at least 19 bags of chocolate?

$\left\lceil \frac{n}{32} \right\rceil = (19-1) \quad n = 18 \times 32 + 1 = \boxed{577}$

(iv) There are 920 positive integers such that each integer is of form 6k for some integer $k \in Z$. Then there are at least m integers out of the 920 numbers, say, $n_1, \ldots, n_m$ such that $n_1 \ (mod\ 9) = n_2 \ (mod\ 9) = \cdots = n_m \ (mod\ 9)$. What is the best value of m?

3, 0, 6

D: 920 numbers

C: $\{0, 3, 6\}$

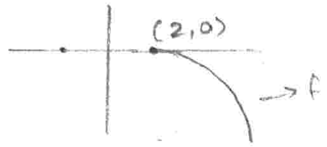
$m = \left\lceil \frac{920}{3} \right\rceil = \lceil 306.66 \rceil = 307$

$1 \to 6$
$2 \to 3$
$3 \to 0$
$4 \to 6$

QUESTION 7. (6 points) Let $f : [2, \infty[\ \to \ ]-\infty, 0]$ be a function such that $f(x) = -\sqrt{x-2}$.

(i) By drawing, is f one-to-one and onto?

since Range = Codomain
so onto



Since it passes horizontal line test → it is 1-1

(ii) If the answer to (i) is yes, find the domain and the co-domain of $f(x)$, then find $f^{-1}(x)$.

Domain = $]-\infty, 0]$

Codomain = $[2, \infty[$

$y = -\sqrt{x-2}$

$x = -\sqrt{y-2}$

$(-x)^2 = y - 2$

$y = x^2 + 2$

$\boxed{f^{-1}(x) = x^2 + 2}$

QUESTION 8. (6 points) Use math induction and prove that $14 \mid (13^{(2n)} - 1)$ for every positive integer $n \geq 1$.

Step 1: For $n = 1$, $13^2 - 1 = 168 = 14 \times 12$

→ so $14 \mid 13^2 - 1$

Step 2: Assume $(14 \mid 13^{2n} - 1)$ is true for some +ve integer $n \geq 1$

Step 3: To prove: $14 \mid 13^{2n+2} - 1$

$13^{2n} \cdot 13^2 - 1 = 13^{2n} \cdot 13^2 - 13^2 + 13^2 - 1$

$= 13^2 \underbrace{(13^{2n} - 1)}_{\text{divisible by 14, step \#2}} + \underbrace{(13^2 - 1)}_{\text{divisible by 14, step \#1}}$

$= 13^2 (14m) + 14(12)$, for some integer m

$\Rightarrow 14 \mid 13^{2n+2} - 1$

$\Rightarrow 14 \mid 13^{2n} - 1$ for every +ve integer n

QUESTION 9. (6 points)

(i) Let $A = \{2, 4, 5, 7\}$. Define " = " on A such that for every $a, b \in A$, $a" = "b$ iff $b = ak$ for some integer $k \neq 0$. Convince me that " = " is not an equivalence relation.

Let $a = 2$ & $b = 4$.

$b = ak$, i.e., $4 = 2k$ for some integer k

But $a \neq bk$ for some integer k. So, it is not symmetric. Hence not an equivalence relation

(ii) Let $A = \{2, -2, 7, -7, 9, -9, 11\}$. Define " = " on A such that for every $a, b \in A$, $a" = "b$ iff $b = ak$ for some integer $k \neq 0$. Then " = " is an equivalence relation. Find all distinct equivalence classes.

$[2] = \{2, -2\}$ $\quad$ $[11] = \{11\}$

$[7] = \{7, -7\}$

$[9] = \{9, -9\}$

QUESTION 10. (6 points) Let $a_n = 2a_{n-1} + 35a_{n-2} + 24(3^n)$. Find a general formula for a_n, no need to find c_1, c_2.

$$a_n - 2a_{n-1} - 35a_{n-2} = 24(3^n)$$

① Find a_h (homogeneous part)

$$a_n - 2a_{n-1} - 35a_{n-2} = 0$$

$$[\alpha^n - 2\alpha^{n-1} - 35\alpha^{n-2} = 0] \div \alpha^{n-2}$$

$$\alpha^2 - 2\alpha - 35 = 0$$

$$\alpha^2 - 7\alpha + 5\alpha - 35 = 0$$

$$\alpha(\alpha - 7) + 5(\alpha - 7) = 0$$

$$\alpha = 7, -5$$

$$a_h(n) = c_1(7)^n + c_2(-5)^n$$

② $a_p(n) = A(3^n)$ by comparing with $24(3^n)$

To find A: $a_p(n) - 2a_p(n-1) - 35a_p(n-2) = 24(3^n)$

$$[A(3^n) - 2A(3^{n-1}) - 35(A\,3^{n-2}) = 24(3^n)] \div 3^n$$

$$A - \frac{2A}{3} - \frac{35A}{9} = 24$$

$$A\left(\frac{-32}{9}\right) = 24 \Rightarrow A = \frac{-27}{4}$$

So: $a_n = a_h(n) + a_p(n) = c_1(7)^n + c_2(-5)^n - \frac{27}{4}(3^n)$